STRUCTURES OF DISTRIBUTIONS OF BINARY SEMI-ISOLATING FORMULAS: (1) DETERMINISTIC AND ABSORBING STRUCTURES, (2) STRUCTURES FOR ACYCLIC GRAPHS (with E.V. Ovchinnikova)

S.V. Sudoplatov

Sobolev Institute of Mathematics,
Novosibirsk State Technical University,
Novosibirsk State University

9th Panhellenic Logic Symposium
Athens, July 16, 2013

Semi-isolation and isolation

Definition (A. Pillay). Let $\mathcal{M}$ be a model of a theory T, $\bar{a}$ and $\bar{b}$ be tuples in $\mathcal{M}$, A be a subset of M. The tuple $\bar{a}$ *semi-isolates* the tuple $\bar{b}$ over the set A if there exists a formula $\varphi(\bar{a}, \bar{y}) \in \mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a}, \bar{y}) \vdash \mathrm{tp}(\bar{b}/A)$ holds.

In this case we say that the formula $\varphi(\bar{a}, \bar{y})$ (with parameters in A) *witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A*.

Similarly, a tuple $\bar{a}$ *isolates* a tuple $\bar{b}$ over A if there exists a formula $\varphi(\bar{a},\bar{y})\in\mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a},\bar{y})\vdash\mathrm{tp}(\bar{b}/A)$ and $\varphi(\bar{a},\bar{y})$ is a principal (i. e., isolating) formula.

In this case we say that the formula $\varphi(\bar{a},\bar{y})$ (with parameters in A) *witnesses that $\bar{b}$ is isolated over $\bar{a}$ with respect to A*.

If $\bar{a}$ (semi-)isolates $\bar{b}$ over $\varnothing$, we simply say that $\bar{a}$ (*semi-*)*isolates* $\bar{b}$; and if a formula $\varphi(\bar{a},\bar{y})$ witnesses that $\bar{a}$ (semi-)isolates $\bar{b}$ over $\varnothing$ then we say that $\varphi(\bar{a},\bar{y})$ *witnesses that* $\bar{a}$ (*semi-*)*isolates* $\bar{b}$.

Semi-isolation and isolation

For a family $R \subset S(T)$ of 1-types we denote by I_R (in $\mathcal{M}$) the set

$$\{(a, b) \mid \mathrm{tp}(a), \mathrm{tp}(b) \in R \text{ and } a \text{ isolates } b\}$$

and by SI_R (in $\mathcal{M}$) the set

$$\{(a, b) \mid \mathrm{tp}(a), \mathrm{tp}(b) \in R \text{ and } a \text{ semi-isolates } b\}.$$

Clearly, $I_R \subseteq \mathrm{SI}_R$ and, for any set of realizations of types in R, the relations I_R and SI_R are reflexive. The relation of semi-isolation on the set of tuples in an arbitrary model is transitive and, in particular, any relation SI_R is transitive. At the same time I_R may be non-transitive.

$$I_p \rightleftharpoons I_{\{p\}}, \ \ \mathrm{SI}_p \rightleftharpoons \mathrm{SI}_{\{p\}}.$$

Principal arcs and edges

Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\varnothing)$, realized in $\mathcal{M}$, and all (p, q)-*preserving* (p, q)-*semi-isolating*, $(p \rightarrow q)$-, or $(q \leftarrow p)$-*formulas* $\varphi(x, y)$ of T, i. e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a, y) \vdash q(y)$. Now, for each such a formula $\varphi(x, y)$, we define a binary relation $R_{p,\varphi,q} \rightleftharpoons \{(a, b) \mid \mathcal{M} \models p(a) \wedge \varphi(a, b)\}$. If $(a, b) \in R_{p,\varphi,q}$, (a, b) is called a (p, φ, q)-*arc*. If $\varphi(a, y)$ is principal (over a), the (p, φ, q)-arc (a, b) is also *principal*.

Principal arcs and edges

If $\varphi(x, y)$ is $(p \leftrightarrow q)$-formula, i.e., both $(p \rightarrow q)$- and $(q \rightarrow p)$-formula, and $\models p(a) \cup \{\varphi(a, b)\} \cup q(b)$ then set $[a, b] \rightleftharpoons \{(a, b), (b, a)\}$ is said to be a (p, φ, q)-*edge*. If the (p, φ, q)-edge $[a, b]$ consists of principal (p, φ, q)- and $(q, \varphi(y, x), p)$-arcs then $[a, b]$ is a *principal* (p, φ, q)-edge. (p, φ, q)-arcs and (p, φ, q)-edges are called *arcs* and *edges* respectively if we say about fixed or some formula $\varphi(x, y)$. If (a, b) is a principal arc and (b, a) is not a principal arc (on any formula) then (a, b) is called *irreversible*.

$\mathrm{PF}(p, q)$

For types $p(x), q(y) \in S(\varnothing)$, we denote by $\mathrm{PF}(p, q)$ the set

$$\{\varphi(x, y) \mid \varphi(a, y) \text{ is a principal formula, } \varphi(a, y) \vdash q(y), \text{ where } \models p(a)\}.$$

Let $\mathrm{PE}(p, q)$ be the set of pairs of formulas $(\varphi(x, y), \psi(x, y)) \in \mathrm{PF}(p, q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a, y)$ and $\psi(a, y)$ coincide. Clearly, $\mathrm{PE}(p, q)$ is an equivalence relation on the set $\mathrm{PF}(p, q)$. Notice that each $\mathrm{PE}(p, q)$-class E corresponds to either a principal edge or to an irreversible principal arc connecting realizations of p and q by some (any) formula in E. Thus the quotient $\mathrm{PF}(p, q)/\mathrm{PE}(p, q)$ is represented as a disjoint union of sets $\mathrm{PFS}(p, q)$ and $\mathrm{PFN}(p, q)$, where $\mathrm{PFS}(p, q)$ consists of $\mathrm{PE}(p, q)$-classes corresponding to principal edges and $\mathrm{PFN}(p, q)$ consists of $\mathrm{PE}(p, q)$-classes corresponding to irreversible principal arcs.

Labels for isolating formulas

Let T be a complete theory without finite models, $U = U^- \dot\cup \{0\} \dot\cup U^+$ be an alphabet of cardinality $\geq |S(T)|$ and consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$.[1] The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of U are called *labels*.

[1] If U is at most countable, we assume that U is a subset of the set $\mathbb{Z}$ of integers.

Labels for isolating formulas and labelling functions

Let $\nu(p,q)$: $\mathrm{PF}(p,q)/\mathrm{PE}(p,q) \to U$ be injective *labelling functions*, $p(x)$, $q(y) \in S(\varnothing)$, for which negative elements correspond to the classes in $\mathrm{PFN}(p,q)/\mathrm{PE}(p,q)$ and non-negative elements correspond to the classes in $\mathrm{PFS}(p,q)/\mathrm{PE}(p,q)$ such that 0 is defined only for $p=q$ and is represented by the formula $(x \approx y)$, $\nu(p) \rightleftharpoons \nu(p,p)$. We additionally assume that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ (where, as usual, we denote by ρ_f the image of the function f) and $\rho_{\nu(p,q)} \cap \rho_{\nu(p',q')} = \varnothing$ if $p \neq q$ and $(p,q) \neq (p',q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Below we shall consider only regular labelling functions and their regular families. We denote by $\theta_{p,u,q}(x,y)$ formulas in $\mathrm{PF}(p,q)$ with a label $u \in \rho_{\nu(p,q)}$. If the type p is fixed and $p=q$ then the formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.

Algebra of distributions for binary isolating formulas

For types $p_1, p_2, \ldots, p_{k+1} \in S^1(\varnothing)$ and sets $X_1, X_2, \ldots, X_k \subseteq U$ of labels we denote by

$$P(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$

the set of all labels $u \in U$ corresponding to formulas $\theta_{p_1,u,p_{k+1}}(x, y)$ satisfying, for realizations a of p_1 and some $u_1 \in X_1, \ldots, u_k \in X_k$, the following condition:

$$\theta_{p_1,u,p_{k+1}}(a, y) \vdash \theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(a, y),$$

where

$$\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x, y) \rightleftharpoons$$
$$\rightleftharpoons \exists x_2, x_3, \ldots, x_{k-1}, x_k(\theta_{p_1,u_1,p_2}(x, x_2) \wedge \theta_{p_2,u_2,p_3}(x_2, x_3) \wedge \ldots$$
$$\ldots \wedge \theta_{p_{k-1},u_{k-1},p_k}(x_{k-1}, x_k) \wedge \theta_{p_k,u_k,p_{k+1}}(x_k, y)).$$

Thus the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra of distributions of binary isolating formulas* with k-ary operations

$$P(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),$$

Algebra of distributions for binary isolating formulas

If each set X_i is a singleton consisting of an element u_i then we use u_i instead of X_i in $P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ and write

$$P(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}).$$

If all types p_i equal to a type p then we write $P_p(X_1, X_2, \dots, X_k)$ and $P_p(u_1, u_2, \dots, u_k)$ as well as $\lfloor X_1, X_2, \dots, X_k \rfloor_p$ and $\lfloor u_1, u_2, \dots, u_k \rfloor_p$ instead of

$$P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$$

and

$$P(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$$

respectively. We omit the index $\cdot_p$ if the type p is fixed. In this case, we write $\theta_{u_1, u_2, \dots, u_k}(x, y)$ instead of $\theta_{p, u_1, p, u_2, \dots, p, u_k, p}(x, y)$.

Labelling groupoid

THEOREM (I.V.Shulepov – S.)

For any complete theory T, any type $p \in S(T)$ having the model $\mathcal{M}_p$, and the regular labelling function $\nu(p)$, any operation $P_p(\cdot, \cdot, \ldots, \cdot)$ on the set $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$ is interpretable by a term of the groupoid $\mathfrak{P}_{\nu(p)} \rightleftharpoons \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}; \lfloor \cdot, \cdot \rfloor \rangle$. The groupoid $\mathfrak{P}_{\nu(p)}$ has the unit $\{0\}$ and, having right associativity, is a monoid.

Deterministic and almost deterministic groupoids

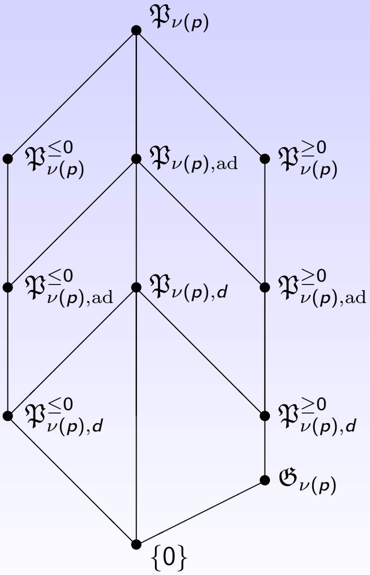
A structure $\mathfrak{P}_{\nu(p)}$ is called (*almost*) *deterministic* if the set $\lfloor u_1, u_2 \rfloor$ is a singleton (is nonempty and finite) for any $u_1, u_2 \in \rho_{\nu(p)}$. Any deterministic structure $\mathfrak{P}_{\nu(p)}$ is a monoid (being almost deterministic). It is generated by the monoid $\mathfrak{P}'_{\nu(p)} = \langle \rho_{\nu(p)}; \odot \rangle$, where $\lfloor u, v \rfloor = \{u \odot v\}$ for $u, v \in \rho_{\nu(p)}$.

Hasse diagram

A Hasse diagram is presented in Figure 1 illustrating the links of the structure $\mathfrak{P}_{\nu(p)}$ with structures above, being restrictions of $\mathfrak{P}_{\nu(p)}$ to subalphabets of U. Here the superscripts $\cdot^{\leq 0}$ and $\cdot^{\geq 0}$ point out on restrictions of $\mathfrak{P}_{\nu(p)}$ to the sets of non-positive and non-negative elements respectively, the subscripts $\cdot_d$ and $\cdot_{\rm ad}$ indicate the sets of deterministic and almost deterministic elements; $\mathfrak{G}_{\nu(p)}$ is a submonoid of monoid $\mathfrak{P}_{\nu(p),d}$ consisting of all non-negative deterministic elements u in $\rho_{\nu(p)}$, for which u^{-1} are also deterministic;
the monoid $(\mathfrak{G}_{\nu(p)})'$ is a group.

Just $\mathfrak{P}_{\nu(p)}$ and $\mathfrak{P}^{\leq 0}_{\nu(p)}$ may not be monoids.

Hasse diagram



I-groupoids

Let $U = U^- \dot\cup \{0\} \dot\cup U^+$ be an alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, and zero 0. As above we write $u < 0$ for any element $u \in U^-$, $u > 0$ for any element $u \in U^+$, and $u \cdot v$ instead of $\{u\} \cdot \{v\}$ considering an operation $\cdot$ on the set $\mathcal{P}(U) \setminus \{\varnothing\}$.

I-groupoids

A groupoid $\mathfrak{P} = \langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle$ is called an *I-groupoid* if it satisfies the following conditions:

• the set $\{0\}$ is the unit of the groupoid $\mathfrak{P}$;

• the operation $\cdot$ of the groupoid $\mathfrak{P}$ is generated by the function $\cdot$ on elements in U such that every elements $u, v \in U$ define a nonempty set $(u \cdot v) \subseteq U$: for any sets $X, Y \in \mathcal{P}(U) \setminus \{\varnothing\}$ the following equality holds:

$$X \cdot Y = \bigcup \{x \cdot y \mid x \in X, y \in Y\};$$

• if $u < 0$ then the sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;

I-groupoids

- if $u > 0$ and $v > 0$ then the set $u \cdot v$ consists of non-negative elements;
- for any $u > 0$ there is the unique *inverse* element $u^{-1} > 0$ such that $0 \in (u \cdot u^{-1}) \cap (u^{-1} \cdot u)$;
- if a positive element u belongs to a set $v_1 \cdot v_2$ then u^{-1} belongs to $v_2^{-1} \cdot v_1^{-1}$;
- for any elements $u_1, u_2, u_3 \in U$ the following inclusion holds:

$$(u_1 \cdot u_2) \cdot u_3 \supseteq u_1 \cdot (u_2 \cdot u_3),$$

and the strict inclusion

$$(u_1 \cdot u_2) \cdot u_3 \supset u_1 \cdot (u_2 \cdot u_3)$$

may be satisfied only for $u_1 < 0$ and $|u_2 \cdot u_3| \geq \omega$;

• the groupoid $\mathfrak{P}$ contains the *deterministic* subgroupoid $\mathfrak{P}_d^{\geq 0}$ (being a monoid) with the universe $\mathcal{P}(U_d^{\geq 0}) \setminus \{\varnothing\}$, where

$$U_d^{\geq 0} = \{u \in U^{\geq 0} \mid u^{-1} \cdot u = \{0\}\};$$

any set $u \cdot v$ is a singleton for $u, v \in U_d^{\geq 0}$.

By the definition each I-groupoid $\mathfrak{P}$ contains I-subgroupoids $\mathfrak{P}^{\leq 0}$ and $\mathfrak{P}^{\geq 0}$ with the universes $\mathcal{P}(U^{-} \cup \{0\}) \setminus \{\varnothing\}$ and $\mathcal{P}(U^{+} \cup \{0\}) \setminus \{\varnothing\}$ respectively. The structure $\mathfrak{P}^{\geq 0}$ is a monoid.

I-groupoids

THEOREM (I.V.Shulepov – S.)

For any (at most countable) I-groupoid $\mathfrak{P}$, there is a (small) theory T with a type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}$.

Partial groupoids

The results are generalized for arbitrary family of 1-types forming partial groupoids of isolating formulas.

$\mathrm{SICF}(p, q)$

For types $p(x), q(y) \in S(\varnothing)$, we denote by $\mathrm{SICF}(p, q)$ the set of $(p \to q)$-formulas $\varphi(x, y)$ such that $\{\varphi(a, y)\}$ is consistent for $\models p(a)$. Let $\mathrm{SICE}(p, q)$ be the set of pairs of formulas $(\varphi(x, y), \psi(x, y)) \in \mathrm{SICF}(p, q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a, y)$ and $\psi(a, y)$ coincide. Clearly, $\mathrm{SICE}(p, q)$ is an equivalence relation on the set $\mathrm{SICF}(p, q)$. Notice that each $\mathrm{SICE}(p, q)$-class E corresponds to either a set of (p, φ, q)-edges, or a set of irreversible (p, φ, q)-arcs, or simultaneously a set of (p, φ, q)-edges and of irreversible (p, φ, q)-arcs linking realizations of p and q by any (some) formula φ in E. Thus the quotient $\mathrm{SICF}(p, q)/\mathrm{SICE}(p, q)$ is represented as a disjoint union of sets $\mathrm{SICFE}(p, q)$, $\mathrm{SICFA}(p, q)$, and $\mathrm{SICFM}(p, q)$, where $\mathrm{SICFE}(p, q)$ consists of $\mathrm{SICE}(p, q)$-classes corresponding to sets of edges, $\mathrm{SICFA}(p, q)$ consists of $\mathrm{SICE}(p, q)$-classes corresponding to sets of irreversible arcs, and $\mathrm{SICFM}(p, q)$ consists of $\mathrm{SICE}(p, q)$-classes corresponding to sets containing edges and irreversible arcs.

Labels for semi-isolating formulas

Let T be a complete theory without finite models, $U = U^- \dot\cup \{0\} \dot\cup U^+ \dot\cup U'$ be an alphabet of cardinality $\geq |S(T)|$ and consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, *neutral elements* $u' \in U'$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$.

Labels for semi-isolating formulas and labelling functions

Let $\nu(p,q)$: $\mathrm{SICF}(p,q)/\mathrm{SICE}(p,q) \to U$ be injective *labelling functions*, $p(x), q(y) \in S(\varnothing)$, for which negative elements correspond to the classes in $\mathrm{SICFA}(p,q)/\mathrm{SICE}(p,q)$, positive elements and 0 correspond to the classes in $\mathrm{SICFE}(p,q)/\mathrm{SICE}(p,q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, and neutral elements code the classes in $\mathrm{SICFM}(p,q)/\mathrm{SICE}(p,q)$, $\nu(p) \rightleftharpoons \nu(p,p)$. We additionally assume that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ where, as usual, we denote by ρ_f the image of the function f) and $\rho_{\nu(p,q)} \cap \rho_{\nu(p',q')} = \varnothing$ if $p \neq q$ and $(p,q) \neq (p',q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Below we shall consider only regular labelling functions and their regular families.

Labels for semi-isolating formulas and labelling functions

The labels, corresponding to isolating formulas, are said to be *isolating* whereas each label in $\bigcup\limits_{p,q\in S^1(\varnothing)}\rho_{\nu(p,q)}$ is *semi-isolating*. By the definition, each isolating label belongs to $U^-\,\dot\cup\,\{0\}\,\dot\cup\,U^+$, i. e., it is not neutral.
We denote by $\theta_{p,u,q}(x,y)$ formulas in $\mathrm{SICF}(p,q)$ with a label $u\in\rho_{\nu(p,q)}$. If the type p is fixed and $p=q$ then the formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.

Algebra of distributions for binary semi-isolating formulas

Similarly algebras of distributions for binary isolating formulas, the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra* $\mathfrak{A}$ *of distributions of binary semi-isolating formulas* with k-ary operations

$$\mathrm{SI}(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),$$

where $p_1, \ldots, p_{k+1} \in S^1(\varnothing)$. This algebra has a natural restriction to any family $R \subseteq S^1(\varnothing)$ as well as to the algebras of distributions of binary *isolating* formulas.

Preordered algebra of distributions for binary semi-isolating formulas

For the set U of labels in the algebra $\mathfrak{A}$ of binary semi-isolating formulas of theory T, we define the following relation $\trianglelefteq$: if $u, v \in U$ then $u \trianglelefteq v$ if and only if $u = v$, or $u, v \in \rho_{\nu(p,q)}$ for some types $p, q \in S^1(\varnothing)$ and $\theta_{p,u,q}(a, y) \vdash \theta_{p,v,q}(a, y)$ for some (any) realization a of p. If $u \trianglelefteq v$ and $u \neq v$ we write $u \triangleleft v$.

POSTC-algebra

The preordered algebra $\langle\mathfrak{A};\trianglelefteq\rangle$ equipped with binary operations $(p,(\cdot\,\tau\,\cdot),q)$, $\tau\in\{\vee,\wedge,\circ\}$, and $(p,(\cdot\,\wedge\,\neg\,\cdot),q)$, $p,q\in S^1(\varnothing)$, is called a *preordered algebra with relative set-theoretic operations and the composition* or briefly a POSTC-*algebra*.

We denote by $\mathfrak{M}_{\nu(R)}$ the restriction of POSTC-algebra to the set of labels for a non-empty family R of 1-types.

For triples (p, u, q), where $p, q \in S^1(\varnothing)$, $u \in U \cup \{\varnothing\}$, we define inductively the *rank* $\mathrm{si}(p, u, q)$ *of semi-isolation*:
(1) $\mathrm{si}(p, u, q) = 0$ if $u \notin \rho_{\nu(p,q)}$;
(2) $\mathrm{si}(p, u, q) \geq 1$ if $u \in \rho_{\nu(p,q)}$;
(3) for a positive ordinal α, $\mathrm{si}(p, u, q) \geq \alpha + 1$ if there is a set $\{v_i \mid i \in \omega\}$ of pairwise inconsistent labels such that $v_i \lhd u$ and $\mathrm{si}(p, v_i, q) \geq \alpha$, $i \in \omega$;
(4) for a limit ordinal α, $\mathrm{si}(p, u, q) \geq \alpha$ if $\mathrm{si}(p, u, q) \geq \beta$ for any $\beta \in \alpha$.
As usual, we write $\mathrm{si}(p, u, q) = \alpha$ if $\mathrm{si}(p, u, q) \geq \alpha$ and $\mathrm{si}(p, u, q) \not\geq \beta$ for $\alpha \in \beta$; $\mathrm{si}(p, u, q) \rightleftharpoons \infty$ if $\mathrm{si}(p, u, q) \geq \alpha$ for any ordinal α.

Ranks and degrees of semi-isolation

If types p and q are fixed, we write $\mathrm{si}(u)$ instead of $\mathrm{si}(p,u,q)$ and this value is said to be the *rank of semi-isolation* or the si-*rank* of the label u or of the element $u=\varnothing$ (with respect to the pair (p,q)). For a formula $\theta_{p,u,q}(x,y)$ we set $\mathrm{si}(\theta_{p,u,q}(x,y))\rightleftharpoons\mathrm{si}(u)$.

Clearly, if the theory is small then the si-rank of any label is an ordinal (having a label u with $\mathrm{si}(p,u,q)=\infty$, we get continuum many complete types $r(x,y)\supset p(x)\cup q(y)$).

PROPOSITION

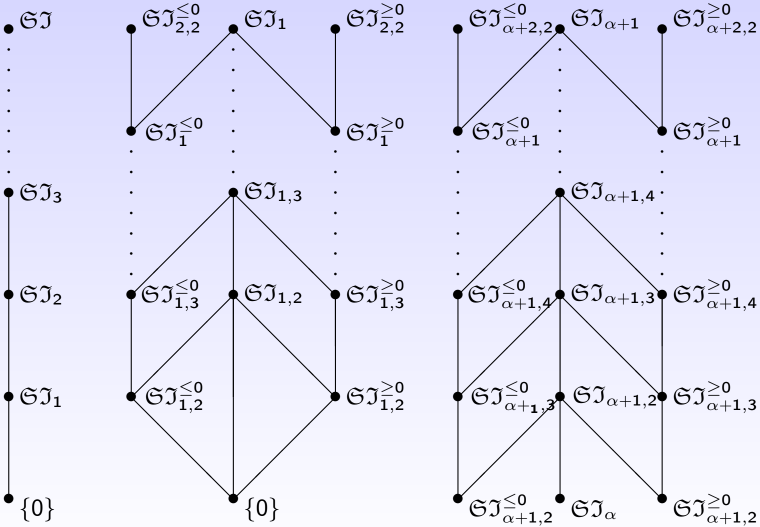
Each si-*rank in a theory* T *is either equal to* ∞ *or less than* $\min\{|T|^+,(\mathrm{MR}(x\approx x)+1)^+\}$. *If Morley rank* $\mathrm{MR}(x\approx x)$ *is equal to an ordinal* α *then any* si-*rank in* T *is not more than* $\alpha+1$.

Similarly Morley degree we define degrees of semi-isolation for labels.

Hierarchy of structures

In the following Figure, the fragments of Hasse diagram are presented illustrating the links of the structure $\mathfrak{S}\mathfrak{I} \rightleftharpoons \mathfrak{S}\mathfrak{I}_{\nu(p)}$ with structures above, being restrictions of $\mathfrak{S}\mathfrak{I}$ to subalphabets of U. Here the superscripts $\cdot^{\leq 0}$ and $\cdot^{\geq 0}$ point out on restrictions of $\mathfrak{S}\mathfrak{I}$ to the sets $U^{\leq 0}$ and $U^{\geq 0}$ respectively, and the subscripts to the upper estimates for si-ranks and si-degrees of labels. In Figure 1, a, a hierarchy of structures $\mathfrak{S}\mathfrak{I}_\alpha$, $\alpha \leq \mathrm{si}(p)$, is depicted starting with the trivial substructure; in Figure 1, b, links between substructures of $\mathfrak{S}\mathfrak{I}_{\nu(p),1}$ are presented; in Figure 1, c, links between substructures of $\mathfrak{S}\mathfrak{I}_{\alpha+1}$ for $1 \leq \alpha < \mathrm{si}(p)$ are shown. For a limit ordinal $\beta \leq \mathrm{si}(p)$, the Hasse diagram for substructures of $\mathfrak{S}\mathfrak{I}_\beta$ is obtained by union of presented diagrams for $\alpha < \beta$. If an ordinal $\beta \leq \mathrm{si}(p)$ is not limit, the Hasse diagram corresponds to the union of presented diagrams for $\alpha < \beta$ with the removal of structures $\mathfrak{S}\mathfrak{I}^{\leq 0}_{\beta+1,2}$ and $\mathfrak{S}\mathfrak{I}^{\geq 0}_{\beta+1,2}$.

Hierarchy of structures



POSTC-monoids

Similarly I-groupoids we axiomatize the class of POSTC-monoids producing binary semi-isolating structures for 1-types and for families of 1-types.

THEOREM

For any (at most countable and having an ordinal $\sup\{\mathrm{si}(u) \mid u \in U\}$) POSTC-monoid $\mathfrak{M}$ there is a (small) theory T with a type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{M}_{\nu(p)} = \mathfrak{M}$.

Absorbing and almost absorbing structures

Now we define some opposite cases to determinacies.
An $I_{\mathcal{R}}$-structure $P_{\nu(R)}$ is *n-absorbing*, for $n \in \omega \setminus \{0\}$, if whenever $u_1, \ldots, u_n$ are nonzero labels in $\rho_{\nu(p_1,p_2)}, \ldots, \rho_{\nu(p_n,p_{n+1})}$ respectively, $p_1, \ldots, p_{n+1} \in \mathcal{R}$, the following conditions hold:
• if some u_i is negative then $P(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1})$ is equal to the set $\rho^-_{\nu(p_1,p_{n+1})}$ of all negative labels in $\rho_{\nu(p_1,p_{n+1})}$;
• if all u_i are positive then $P(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1})$ contains the set $\rho^+_{\nu(p_1,p_{n+1})}$ of all positive labels in $\rho_{\nu(p_1,p_{n+1})}$ (i. e., $P(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) = \rho^+_{\nu(p_1,p_{n+1})}$ or $P(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) = \rho^+_{\nu(p_1,p_{n+1})} \cup \{0\}$).

Absorbing and almost absorbing structures

An $I_{\mathcal{R}}$-structure $P_{\nu(R)}$ is *almost n-absorbing*, for $n \in \omega \setminus \{0\}$, if whenever $u_1, \ldots, u_n$ are nonzero labels in $\rho_{\nu(p_1,p_2)}, \ldots, \rho_{\nu(p_n,p_{n+1})}$ respectively, $p_1, \ldots, p_{n+1} \in \mathcal{R}$, the following conditions hold:

• if some u_i is negative then
$P(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) \setminus \rho^-_{\nu(p_1,p_{n+1})}$ is finite;

• if all u_i are positive then
$P(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) \setminus \rho^+_{\nu(p_1,p_{n+1})}$ is finite.

Absorbing and almost absorbing structures

A $\mathrm{POSTC}_{\mathcal{R}}$-structure $\mathfrak{M}$ is *n-absorbing*, for $n \in \omega \setminus \{0\}$, if whenever $u_1, \ldots, u_n$ are nonzero labels in $\rho_{\nu(p_1,p_2)}, \ldots, \rho_{\nu(p_n,p_{n+1})}$ respectively, $p_1, \ldots, p_{n+1} \in \mathcal{R}$, the following conditions hold:

- if some u_i is negative then $\mathrm{SI}(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1})$ is equal to the set $\rho^-_{\nu(p_1,p_{n+1})}$ of all negative labels in $\rho_{\nu(p_1,p_{n+1})}$;
- if all u_i are positive then $\mathrm{SI}(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1})$ contains the set $\rho^+_{\nu(p_1,p_{n+1})}$ of all negative labels in $\rho_{\nu(p_1,p_{n+1})}$;
- if the labels u_i are positive or belong to U' and some u_i belongs to U' then $\mathrm{SI}(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1})$ contains the set $(\rho^+_{\nu(p_1,p_{n+1})})'$ of all labels of $U^+ \cup U'$ laying in $\rho_{\nu(p_1,p_{n+1})}$.

Absorbing and almost absorbing structures

A $\mathrm{POSTC}_{\mathcal{R}}$-structure $\mathfrak{M}$ is *almost n-absorbing*, for $n \in \omega \setminus \{0\}$, if whenever $u_1, \ldots, u_n$ are nonzero labels in $\rho_{\nu(p_1,p_2)}, \ldots, \rho_{\nu(p_n,p_{n+1})}$ respectively, $p_1, \ldots, p_{n+1} \in \mathcal{R}$, the following conditions hold:

- if some u_i is negative then $\mathrm{SI}(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) \setminus \rho^-_{\nu(p_1,p_{n+1})}$ is finite;
- if all u_i are positive then $\mathrm{SI}(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) \setminus \rho^+_{\nu(p_1,p_{n+1})}$ is finite;
- if the labels u_i are positive or belong to U' and some u_i belongs to U' then $\mathrm{SI}(p_1, u_1, p_2, u_2, \ldots, u_n, p_{n+1}) \setminus (\rho^-_{\nu(p_1,p_{n+1})})'$ is finite.

Hierarchy of absorbing structures

PROPOSITION

For all $n \in \omega \setminus \{0\}$, if an associative structure $\mathfrak{M}$ is (almost) n-absorbing then $\mathfrak{M}$ is (almost) $(n+1)$-absorbing.

Hierarchy of absorbing structures

Now we denote by $\mathrm{AbI}_{\mathcal{R},n}$ ($\mathrm{AbSI}_{\mathcal{R},n}$, $\mathrm{AAbI}_{\mathcal{R},n}$, $\mathrm{AAbSI}_{\mathcal{R},n}$, respectively) the class of associative n-absorbing $I_{\mathcal{R}}$-structures (n-absorbing $SI_{\mathcal{R}}$-structures, almost n-absorbing $I_{\mathcal{R}}$-structures, almost n-absorbing $SI_{\mathcal{R}}$-structures). By Proposition, we have inclusions $\mathrm{AbI}_{\mathcal{R},n} \subseteq \mathrm{AAbI}_{\mathcal{R},n}$, $\mathrm{AbSI}_{\mathcal{R},n} \subseteq \mathrm{AAbSI}_{\mathcal{R},n}$, $\mathrm{AbI}_{\mathcal{R},n} \subseteq \mathrm{AbI}_{\mathcal{R},n+1}$, $\mathrm{AbSI}_{\mathcal{R},n} \subseteq \mathrm{AbSI}_{\mathcal{R},n+1}$, $\mathrm{AAbI}_{\mathcal{R},n} \subseteq \mathrm{AAbI}_{\mathcal{R},n+1}$, $\mathrm{AAbSI}_{\mathcal{R},n} \subseteq \mathrm{AAbSI}_{\mathcal{R},n+1}$, $n \in \omega \setminus \{0\}$.

All these inclusions are strict.

Powerful graphs

Let $\Gamma = \langle X, Q \rangle$ be a graph, and a be a vertex of Γ. Recall that the set $\bigtriangledown_Q(a) = \bigcup\limits_{n\in\omega} Q^n(a,\Gamma)$ (respectively $\bigtriangleup_Q(a) = \bigcup\limits_{n\in\omega} Q^n(\Gamma, a)$) is a *upper* (*lower*) *Q-cone* of a. We call the Q-cones $\bigtriangledown_Q(a)$ and $\bigtriangleup_Q(a)$ by *cones* and denote by $\bigtriangledown(a)$ and $\bigtriangleup(a)$ respectively if Q is fixed.

Powerful graphs

A countable acyclic directed graph $\Gamma = \langle X; Q \rangle$ is said to be *powerful* if the following conditions hold:
(a) the automorphism group of Γ is *transitive*, that is any two vertices are connected by an automorphism;
(b) the formula $Q(x, y)$ is equivalent in the theory $\mathrm{Th}(\Gamma)$ to a disjunction of principal formulas;
(c) $\mathrm{acl}(\{a\}) \cap \triangle_Q(a) = \{a\}$ for each vertex $a \in X$;
(d) $\Gamma \models \forall x, y\, \exists z\, (Q(z, x) \wedge Q(z, y))$ (the *pairwise intersection property*).

Special structures

It is known that powerful graphs as well as, in fact, associated structures $\mathfrak{P}_{\nu(p)}$ play a key role for the constructions of series of Ehrenfeucht theories.

Recall that a monoid $\mathfrak{P}_{\nu(p)}$ is *special* if $\rho_{\nu(p)} \cap U^- \neq \emptyset$ and for any elements $u_1, u_2, \ldots, u_n, v \in \rho_{\nu(p)}$, where $u_1 < 0, \ldots, u_n < 0$, $v \geq 0$, and for any element $u' \in u_1 u_2 \ldots u_n v$ there is an element $v' \geq 0$ such that $u' \in v' u_1 u_2 \ldots u_n$.

A special monoid $\mathfrak{P}_{\nu(p)}$ is called PIP-*special* if each negative element $u \in \rho_{\nu(p)}$ is a PIP-element, i. e., $u \in uv$ for any $v \in \rho_{\nu(p)}$.

Special structures and theories

Having a special monoid (for a special small theory T) the process of construction of a limit model over a type p is reduced to a sequence of θ_{u_n}-extensions, $u_n < 0$, $n \in \omega$, of prime models over realizations of p: for any limit model $\mathcal{M}$ over p there is an elementary chain $(\mathcal{M}(a_n))_{n\in\omega}$, $\models p(\bar{a}_n)$, such that its union forms $\mathcal{M}$ and $\models \theta_{u_n}(a_{n+1}, a_n)$ is satisfied, $n \in \omega$. In this case the isomorphism type of $\mathcal{M}$ is defined by the sequence $(u_n)_{n\in\omega}$.

If a PIP-special monoid exists then, by adding of multiplace predicates, each prime model over a tuple of realizations of p is transformed to a model isomorphic to $\mathcal{M}_p$. Thus, the type p is connected with the unique, up to isomorphism, prime model over realizations of p and with some (finite, countable, or continuum) number of limit models over p, which is defined by some quotient for the set of sequences $(u_n)_{n\in\omega}$, $u_n \in U^- \cap \rho_{\nu(p)}$, $n \in \omega$. The action of these quotients is defined by some identifications $(w \approx w')$ of words in the alphabet $U^- \cap \rho_{\nu(p)}$ such that if $w = u_1 \dots u_m$ and $w' = u'_1 \dots u'_n$ then for any $v \in U^{\geq 0} \cap \rho_{\nu(p)}$ and $u_0 \in u_1 \dots u_m v$ there exists $v' \in U^{\geq 0} \cap \rho_{\nu(p)}$ with $u_0 \in v' u'_1 u'_2 \dots u'_n$.

Let T be a theory with a type p having the model $\mathcal{M}_p$, $\mathfrak{P}_{\nu(p)}$ be an $I_{\nu(p)}$-groupoid, and X be a subset of $\rho_{\nu(p)}$ having a cardinality λ. We say that X is (formula) *definable* if for a realization a of p the set of solutions of $L_{\lambda^+,\omega}$-formula $\varphi(a,y)=\bigvee\limits_{u\in X}\theta_u(a,y)$ in $\mathcal{M}_p$ is $L_{\omega,\omega}$-definable in $\mathcal{M}_p$ by a formula $\psi(a,y)$. In this case we say that the formula $\psi(x,y)$ *witnesses* definability of X.
A groupoid $\mathfrak{P}_{\nu(p)}$ *generates the strict order property* if for some definable set $X\subseteq\rho_{\nu(p)}$, for a witnessing formula $\varphi(x,y)$, and for some realizations a and b of p satisfying $\models\theta_v(b,a)$ with a label $v\in\rho_{\nu(p)}$, the inclusion $\varphi(a,\mathcal{M}_p)\subset\varphi(b,\mathcal{M}_p)$ holds.

Special structures without the strict order property

The following theorem shows that assuming the non-validity of the strict order property (i.e., with NSOP), we can not construct a special monoid $\mathfrak{P}_{\nu(p)}$ being almost deterministic, with bounded cardinalities for products $u_1 \ldots u_m$, or almost absorbing. Hence, these monoids can not be too small or too large with respect to their operations.

THEOREM

If T is a small theory with a type p, and a special monoid $\mathfrak{P}_{\nu(p)}$ is almost deterministic, with a constant C bounding cardinalities of sets $u_1 \ldots u_m$, or almost n-absorbing for some n, then $\mathfrak{P}_{\nu(p)}$ generates the strict order property.

Structures for acyclic graphs

THEOREM (E. V. Ovchinnikova – S.)

If T is a theory of an acyclic graph $\langle M; Q\rangle$ with some unary predicates, a 1-type $p(x)$, and a deterministic algebra $\mathfrak{P}_{\nu(p)}$, then $\mathfrak{P}_{\nu(p)}$ is generated by a free product $*_{i\in I}\mathbb{Z}_i * *_{j\in J}\mathbb{Z}_{2,j} * *_{k\in K}\langle\omega^*_k; +\rangle$ for some copies $\mathbb{Z}_i$ of group $\mathbb{Z}$, copies $\mathbb{Z}_{2,j}$ of group $\mathbb{Z}_2$, and copies $\langle\omega^*_k; +\rangle$ of monoid $\langle\omega^*; +\rangle$. If there are $\langle\omega^*_k; +\rangle$ then the type p is not isolated.

Structures for acyclic graphs

PROPOSITION (E. V. Ovchinnikova – S.)

For any theory T of an acyclic graph with bounded diameter and with unary predicates, for a nonempty family R of types in $S^1(T)$ and a regular family $\nu(R)$ of labelling functions, the structure $\mathfrak{P}_{\nu(R)}$ is almost deterministic.

Examples

I. If $|\rho_{\nu(p)}| = 1$ then $(x \approx y)$ is the unique principal formula up to equivalence. It is possible only in the following cases:
(1) T is small (i. e., with countable $S(\varnothing)$) and satisfies some of the following condition:
(a) $p(x)$ is a principal type with the only realization;
(b) $p(x)$ is a non-principal type such that if a set $\{\varphi(a, y) \wedge \neg(a \approx y)\} \cup p(y)$ is consistent, where $\varphi(x, y)$ is a formula of T, $\models p(a)$, then $\varphi(a, y) \nvdash p(y)$;
(2) T is a theory with continuum many types and for any formula $\varphi(x, y)$ of T and for a realization a of $p(x)$ if the set $\{\varphi(a, y) \wedge \neg(a \approx y)\} \cup p(y)$ is consistent and $\varphi(a, y) \vdash p(y)$ then there are no isolating formulas $\psi(a, y)$ such that $\psi(a, y) \vdash \varphi(a, y) \wedge \neg(a \approx y)$.
The case 1,a is represented by a type being realized by a constant; the cases 1,b and 2 are represented by theories of unary predicates with non-principal types $p(x)$ and having countably many and continuum many types respectively.

Examples

II. Let $\rho_{\nu(p)} = \{0,1\}$. Then $1^{-1} = 1$ and any realization a of p is linked with the only realization b of p for which $\models \theta_1(a,b)$ and, moreover, $\models \theta_1(b,a)$. Then the set of realizations of p splits on two-element equivalence classes consisting of θ_1-edges. If p is a principal type of a small theory then a θ_1-edge is unique, and if p is non-principal the number of this edges can vary from 1 to the infinity depending on a model of a theory.

III. Let $\rho_{\nu(p)} = \{-1, 0\}$ be a set for a small theory T. By non-symmetric semi-isolation, the type $p(x)$ is non-principal and the formula $\theta_{-1}(x, y)$ witnesses that SI_p is non-symmetric. The formula $\theta_{-1,-1}(x, y) \rightleftharpoons \exists z(\theta_{-1}(x, z) \wedge \theta_{-1}(z, y))$ is also witnessing that SI_p is non-symmetric. By assumption the formula $\theta_{-1,-1}(a, y)$ is equivalent to the formula $\theta_{-1}(a, y)$. It means that, on a set of realizations of p, the relation described by the formula $\theta_{-1}(x, y) \vee (x \approx y)$ is an infinite partial order. This partial order is dense since if the element a has a covering element then the formula $\theta_{-1}(a, y)$ is equivalent to the disjunction of consistent formulas $\theta_{-1}(a, y) \wedge \theta_{-1,-1}(a, y)$ and $\theta_{-1}(a, y) \wedge \neg\theta_{-1,-1}(a, y)$, but it is impossible for the principal formula $\theta_{-1}(a, y)$.

Examples

We consider, as a theory with $\rho_{\nu(p)} = \{-1, 0\}$, the Ehrenfeucht's theory T, i. e. the theory of a structure $\mathcal{M}$, formed from the structure $\langle \mathbb{Q}; < \rangle$ by adding constants c_k, $c_k < c_{k+1}$, $k \in \omega$, such that $\lim\limits_{k\to\infty} c_k = \infty$. The type $p(x)$, isolated by the set of formulas $c_k < x$, $k \in \omega$, has exactly two non-equivalent isolating formulas: $\theta_{-1}(a, y) = (a < y)$ and $\theta_0(a, y) = (a \approx y)$, where $\models p(a)$.

Examples

IV. Let $\rho_{\nu(p)} = \{-1, 0, 1\}$. Realizing this equation, we consider the Ehrenfeucht's example, where each element a is replaced by an $<$-antichain consisting of two elements a' and a'' such that $\models \theta_1(a', a'') \wedge \theta_1(a'', a')$. Then we have the following equations for the type $p(x)$ isolated by the set of formulas $c'_k < x$, $k \in \omega$: $P_p(-1, -1) = P_p(-1, 1) = P_p(1, -1) = \{-1\}$, $P_p(1, 1) = \{0\}$.

Examples

V. The equation $\rho_{\nu(p)} = \{-2, -1, 0\}$ with $P_p(-2, -2) = \{-2\}$ and

$$P_p(-2, -1) = P_p(-1, -2) = P_p(-1, -1) = \{-1\}$$

can be fulfilled by two dense strict orders $<_1$ and $<_2$ on a set of realizations of a non-principal type such that $<_1$ immerses $<_2$: $<_1 \circ <_2 \, = \, <_2 \circ <_1 \, = \, <_1$.

VI. Consider a dense linearly ordered set $\mathcal{M} = \langle \mathbb{Q}, < \rangle$, $T = \mathrm{Th}(\mathcal{M})$, and the unique 1-type p of T. Define a labelling function $\nu(p)$, for which 0 corresponds to the formula $(x \approx y)$, 1 to $(x < y)$, and 2 to $(y < x)$. We have $\rho_{\nu(p)} = \{0, 1, 2\}$, $P_p(1, 2) = P_p(2, 1) = \rho_{\nu(p)}$, $P_p(1, 1) = \{1\}$, $P_p(2, 2) = \{2\}$.

Examples

VII. Take a group $\langle G; *\rangle$ and define, on the set G binary predicates Q_g, $g \in G$, by the following rule:

$$Q_g = \{(a, b) \in G^2 \mid a * g = b\}.$$

If $p(x)$ is a type (of a theory T) realized in any model $\mathcal{M} \models T$ containing G exactly by elements in G connected by definable relations Q_g, then the type p is isolated, the set G is finite, and $\rho_{\nu(p)}$ consists of non-negative elements bijective with elements in G. If $\rho_{\nu(p)}$ consists of non-negative elements, is bijective with G, and the set of realizations of a principal type p is not fixed, then, assuming the smallness of the theory, the set G is infinite and the number of connected components with respect to the relation $Q \rightleftharpoons \bigcup\limits_{g \in G} Q_g$ is not bounded. At last if the type p is not isolated then the number of Q-components on sets of realizations of p is also unbounded although the set G can be finite.

Examples

The Cayley table of the group $\langle G; *\rangle$ defines operations $P_p(\cdot, \ldots, \cdot)$ on the set $\rho_{\nu(p)}$ in accordance with links between the relations Q_g.

Examples

VIII. Applying to a concrete group we consider the structure $\mathcal{M} \rightleftharpoons \langle \mathbb{Z}; s^{(1)} \rangle$ with the unary *successor function* $s\colon \mathbb{Z} \leftrightarrow \mathbb{Z}$, where $s(n) = n + 1$ for each $n \in \mathbb{Z}$. For the unique 1-type p of the theory $\mathrm{Th}(\mathcal{M})$ the set of pairwise non-equivalent formulas $\theta_u(x, y)$ is exhausted by the list: $y \approx \underbrace{s \ldots s}_{n \text{ times}}(x)$ and $x \approx \underbrace{s \ldots s}_{n \text{ times}}(y)$, $n \in \omega$. The set $\rho_{\nu(p)}$ consists of non-negative elements linked by additive group of integers.

Examples

IX. We set $T \rightleftharpoons \mathrm{Th}((\mathbb{Q}; <, c_n, c'_n)_{n\in\omega}$, where $<$ is an ordinary strict order on the set $\mathbb{Q}$ of rationals, constants c_n form a strictly increasing sequence, and constants c'_n form a strictly decreasing sequence, $c_n < c'_n$, $n \in \omega$. The theory T has six pairwise non-isomorphic countable models:

• a prime model with empty set of realizations of type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\} \cup \{x < c'_n \mid n \in \omega\}$;

• a prime model over a realization of $p(x)$, with a unique realization of this type;

• a prime model over a realization of type $q(x, y)$ isolated by the set $p(x) \cup p(y) \cup \{x < y\}$; here the set of realizations of $q(x, y)$ forms a closed interval $[a, b]$;

• three limit models over the type $q(x, y)$, in which the sets of realizations of $q(x, y)$ are intervals of forms $(a, b]$, $[a, b)$, (a, b) respectively.

Examples

Consider the type $q(x,y)$. Taking the formula $\varphi(x_1,x_2,y_1,y_2)$ defined by $x_1 \le y_1 < y_2 \le x_2$ we get

$$\varphi(x_1,x_2,y_1,y_2) \equiv \bigvee_{i=0}^{-3} \theta_i(x_1,x_2,y_1,y_2),$$

where $\theta_0(x_1,x_2,y_1,y_2) = (x_1 \approx y_1 < y_2 \approx x_2)$,
$\theta_{-1}(x_1,x_2,y_1,y_2) = (x_1 < y_1 < y_2 \approx x_2)$,
$\theta_{-2}(x_1,x_2,y_1,y_2) = (x_1 \approx y_1 < y_2 < x_2)$,
$\theta_{-3}(x_1,x_2,y_1,y_2) = (x_1 < y_1 < y_2 < x_2)$. The following Cayley table illustrates the algebra of isolating formulas for $q(x,y)$:

P_q	0	-1	-2	-3
0	0	-1	-2	-3
-1	-1	-1	-3	-3
-2	-2	-3	-2	-3
-3	-3	-3	-3	-3

Examples

X. Consider an arbitrary λ-cube C. It is known that all isolating formulas $\theta_u(a, y)$, linking elements in C, are represented by $d_k(a, y)$, where k is the distance between a and b for $\models d_k(a, b)$. Assuming that each label u is denoted by a natural number, defining that distance, for the unique 1-type p and labels $m, n \in \omega$ the set $P_p(m, n)$ consists of all numbers

$$|m + \sum_{i=1}^{n}(-1)^{\delta_i}|,$$

where each δ_i is equal to 0 or 1. If the cardinality λ is finite then we choose only numbers that do not exceed 2^λ.