Introduction to the Bi-modal Analysis of Knowability

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Is it knowable that ‘Syriza formed a coalition government?’
It depends on how one understands ‘knowable’.
You cannot know false things, but if it were true it would be knowable.
So we might formalise ‘knowable’ as:

\[ F \rightarrow \Diamond KF \]  
(Verificationist Knowability – VK)

For any \( F \), if it is true, then it is knowable.
Problem: But $F \rightarrow \diamondsuit KF$ implies, for any $p$:

$$p \rightarrow Kp$$

(Omniscience – OMN)

Which says that all truths are known.

This the Church-Fitch “Knowability Paradox”

It seems we need a better formalization of claims about knowability.
Analyses of the “knowability paradox” tend to treat $K$ as a non-modal operator.

Almost no approach uses Kripke models to analyse knowability and the paradox.

We propose a bi-modal approach to knowability statements, and argue that it is sufficiently flexible to give us coherent and useful formalizations of knowability claims.

Helps uncover what is wrong with the knowability paradox.
Basic Assumptions
Outline

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The system B
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Knowability Principles

Stable Knowability
Monotonic Knowability
Total Knowability
Total Monotonic Knowability
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▶ Bi-modal logic B is a logic with modalities □/◇ and K.
▶ K is the epistemic modality, informally $KF$ reads as ‘$F$ is known’.
▶ □/◇ is an ‘investigation’ or ‘verification’ modality, representing the possible courses a process of gathering information might take.
▶ □$F$ reads as ‘at all stages of investigation $F$ holds’, while ◇$F$ means ‘at some stage of investigation $F$ holds’
▶ Hence, □$KF$, says ‘at every stage of investigation $F$ is known’.
▶ ◇$KF$ says that ‘at some stage of investigation $F$ is (or becomes) known’
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**Axioms:**

A1. Axioms of classical propositional logic
A2. \(\Box (A \to B) \to (\Box A \to \Box B)\)
A3. \(\Box A \to A\)
A4. \(\Box A \to \Box \Box A\)
A5. \(K(A \to B) \to (KA \to KB)\)
A6. \(KA \to A\)

**Inference Rules:** *Modus Ponens, \(\Box\) Necessitation, \(K\) Necessitation*
A model for B is a quadruple \(< W, R_{\Box}, R_K, V >\).

- \(W\) is a set of states.
- \(R_{\Box}\) a transitive and reflexive relation on \(W\).
- \(R_K\) a reflexive relation on \(W\).
- \(V\) is a mapping from propositional variables to subsets of \(W\).

We can prove:

**Theorem 1.**

\emph{B is sound and complete with respect to the class of B models.}

This is proved via the canonical model construction.
With the framework we can now say what is wrong with VK.

If we add $F \rightarrow \Diamond K F$ to B we get the system B+VK.

B+VK models are B models which have all instances of VK true at each state.

But now the $R_K$ relation is also all knowing, i.e. satisfies:

$$\forall x \forall y (x R_K y \rightarrow x = y)$$

Call a model omniscient if all instances of $F \rightarrow K F$ are true at each state.
**Theorem 2.**
*All models of B+VK are omniscient.*

**Proof.**
Assume some state, $x$, of a B+VK model is such that $x \models F$. Since $R_K$ is all-knowing for any $y$ such that $xR_Ky$ we have that $x = y$, and hence every such $y$ is such that $y \models F$, and so $x \models KF$. Hence $F \rightarrow KF$ holds at every state of a B+VK model. ■
Theorem 3.

*B+VK is complete with respect to the class of B+VK models.*

Proof.

As for B except we need to show in addition that $R_K$ is all-knowing. Assume $R_K$ is not all-knowing to derive a contradiction. Let $\Gamma$ and $\Delta$ be maximal consistent sets in the canonical model. Assume $\Gamma R_K \Delta$ but $\Gamma \neq \Delta$; if $\Gamma \neq \Delta$ then there is at least one formula on which they disagree. Assume $X$ is such a formula, and assume $X \in \Gamma$ but $X \notin \Delta$. Since $F \rightarrow \diamond K F$ is an axiom, $X \rightarrow \diamond K X \in \Gamma$, and so by the Church-Fitch “knowability paradox” proof $X \rightarrow K X \in \Gamma$. Hence $K X \in \Gamma$, and so $X \in \Delta$, which is a contradiction.
Consider the following counter-model for $F \to \Diamond \mathbf{K} F$.

- Arrows represent $R_2$ and ovals $R_\mathbf{K}$.
- Both are reflexive, and $R_2$ is transitive:

B model $\mathcal{M}_1$ in which $F \to \Diamond \mathbf{K} F$ is not valid.
**Theorem 4.**

*Not all instances of* $F \rightarrow \Diamond KF$ *hold in* $\mathcal{M}_1$, *in particular* $\mathcal{M}_1 \not \models p \rightarrow \Diamond Kp$.

**Proof.**

At $u$, $p$ is true, but at $w$ and $v$ it is false. Since $R_K$ is reflexive and $w \not \models Kp$, and since $uR_Kv$ and $u \not \models Kp$ and $v \not \models Kp$, hence there is no state $R_{\Box}$-accessible from $u$ where $Kp$ holds, and hence $u \not \models \Diamond Kp$, so $u \not \models p \rightarrow \Diamond Kp$. Hence not all instances of $F \rightarrow \Diamond KF$ hold in $\mathcal{M}_1$. $\blacksquare$
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- Notice that \( p \) does not stay true from state \( u \) to \( w \).
- Call a proposition *stable*, in a given model, if it satisfies \( F \rightarrow \square F \).
- If \( p \) were stable, we would not have a counter-model.
- If \( p \) were stable it would be knowable.

\[ \begin{array}{c}
\text{B model } \mathcal{M}_1 \text{ in which } \\
F \rightarrow \Box K F \text{ is not valid.}
\end{array} \]
To capture this idea we propose:

$$\Box F \rightarrow \Diamond KF.$$  \hspace{1cm} (Stable Knowability – SK)

If $F$ is stably true, then it is knowable, i.e. all stable truths are knowable.

SK does not entail the omniscience defect.
**Theorem 5.**

\[ \Box F \rightarrow \Diamond KF \nvdash p \rightarrow Kp \]

**Proof.**

In \( \mathcal{M}_1 \) if \( \Box X \) holds at any state then \( X \) holds at \( w \), so \( w \vDash KX \), hence all states have \( \Diamond KX \), hence all instances of SK hold in \( \mathcal{M}_1 \), but \( v \nvdash p \rightarrow Kp \). \( \blacksquare \)

B model \( \mathcal{M}_1 \) in which \( F \rightarrow \Diamond KF \) is not valid.
One might think that knowledge is possible only on the basis of conclusive evidence, i.e. when no possible counter-evidence exists.

To capture this idea we propose

$$\Box F \rightarrow \Diamond \Box KF.$$ (Monotonic Knowability – MK)

If $F$ is stably true then it is knowable indefeasibly.

MK also does not suffer from the omniscience defect.
Theorem 6.
\( \Box F \rightarrow \Diamond \Box KF \not\models p \rightarrow Kp \)

Proof.
Similar to Theorem 5. If \( \Box X \) holds at any state then \( \Diamond \Box KX \) holds at all states, so all instances of MK hold in \( \mathcal{M}_1 \). ■
However, not all propositions are stable.

Interesting as they are, SK and MK have limited application. (Good for mathematical knowability perhaps?)

Speaking generally, it seems that saying a proposition is knowable is to say that its truth is decidable (not necessarily formally) by some means or other.

For non-stable propositions the claims about knowability might be formalised thus:

\[ \Diamond KF \lor \Diamond K\neg F. \quad (Total \ Knowability - TK) \]

This too does not yield omniscience.
**Theorem 7.**
\[ \Diamond KF \lor \Diamond K\neg F \not\Vdash p \rightarrow KP \]

**Proof.**
Again consider \( \mathcal{M}_1 \). \( w \models X \lor \neg X \) so \( w \models KX \lor K\neg X \), hence \( \Diamond KX \lor \Diamond K\neg X \) holds at all states, so all instances of TK hold in \( \mathcal{M}_1 \), but \( w \not\Vdash p \rightarrow KP \). \( \blacksquare \)
Again, one might think that knowledge is indefesible. We can formalize this by:

\[ \Diamond \Box K F \lor \Diamond \Box K \neg F, \ (Total \ Monotonic \ Knowability \ – \ TMK) \]

And again, we can prove TMK does not yield omniscience.
**Theorem 8.**
\(\Diamond \BoxKF \lor \Diamond \Box\neg F \not\models p \to Kp\)

**Proof.**
Similar to Theorem 7. Since
\(w \models \Box KX \lor \Box \neg X\) and \(R_{\Box}\) is reflexive
\(w \models \Diamond \Box X \lor \Diamond \Box \neg X\), hence all
instances of \(\Diamond \Box F \lor \Diamond \Box \neg F\) hold at all
states. ■
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So we see that a bi-modal approach to knowability allows us to formulate different principles.

It also enables us to compare each of these principles in a rigorous way.

The following relations hold between the various knowability principles.
Arrows represent derivability in B, but not converses.
Let us prove that TMK is strictly stronger than TK and MK.
Theorem 9.
\[ \Diamond \Box K \neg F \lor \Diamond \Box K \neg F \vdash_B \Diamond K F \lor \Diamond K \neg F \]

Proof.
1. \[ \Diamond \Box K F \lor \Diamond \Box K \neg F \]
2. \[ \Box K F \rightarrow K F \] Reflexivity
3. \[ \Box (\Box K F \rightarrow K F) \] 2 Nec
4. \[ \Box K \neg F \rightarrow K \neg F \] Reflexivity
5. \[ \Box (\Box K \neg F \rightarrow K \neg F) \] 4 Nec
6. \[ \Box (\Box K F \rightarrow K F) \rightarrow (\Diamond \Box K F \rightarrow \Diamond K F) \] from
   \[ \Box (F \rightarrow G) \rightarrow (\Diamond F \rightarrow \Diamond G) \]
7. \[ \Box (\Box K \neg F \rightarrow K \neg F) \rightarrow (\Diamond \Box K \neg F \rightarrow \Diamond K \neg F) \] from
   \[ \Box (F \rightarrow G) \rightarrow (\Diamond F \rightarrow \Diamond G) \]
8. \[ \Diamond \Box K F \rightarrow \Diamond K F \] 3, 6 MP
9. \[ \Diamond \Box K \neg F \rightarrow \Diamond K \neg F \] 5, 7 MP
10. \[ \Diamond K F \lor \Diamond K \neg F \] 1, 8, 9 by propositional reasoning.
**Theorem 10.**
\(\Diamond KF \lor \Diamond K\neg F \nvdash \Box KF \lor \Box K\neg F\)

**Proof.**
Consider model \(M_2:\)

\[
\begin{array}{ccc}
p & \rightarrow & \neg p \\
\bullet & & \bullet \\
\end{array}
\]

\(M_2, B\) model where \(\Diamond KF \lor \Diamond K\neg F\) holds but \(\Box KF \lor \Box K\neg F\) fails

\(u \vdash X \lor \neg X\) so \(u \vdash CX \lor C\neg X\) and \(u \vdash \Diamond CX \lor \Diamond C\neg X\). Similarly for \(v\), hence at all states TK holds. \(u, v \nvdash \Box Kp, \Box K\neg p\), hence \(u \nvdash \Box Kp\) and \(u \nvdash \Box K\neg p\), hence \(M_2 \nvdash \text{TMK}\).
Theorem 11.
\( \Diamond \Box F \vee \Diamond \Box \neg F \vdash_{B} \Box F \rightarrow \Diamond \Box K F \)

Proof.
1. \( \Diamond \Box K F \vee \Diamond \Box \neg F \)
2. \( \neg \Diamond \Box \neg F \rightarrow \Diamond \Box K F \) from 1 by \( (F \vee G) \rightarrow (\neg F \rightarrow G) \)
3. \( \Box \neg \Box \neg F \rightarrow \Diamond \Box K F \) from 2, \( \Box \neg \) for \( \neg \Diamond \)
4. \( K \neg F \rightarrow \neg F \) Reflexivity
5. \( F \rightarrow \neg K \neg F \) 4 contrapositive
6. \( \Diamond F \rightarrow \Diamond \neg K \neg F \) from 5 by Nec and \( \Box (F \rightarrow G) \rightarrow (\Diamond F \rightarrow \Diamond G) \)
7. \( \Diamond F \rightarrow \neg \Box K \neg F \) 6 \( \Box \neg \) for \( \neg \Diamond \)
8. \( F \rightarrow \Diamond F \) reflexivity
9. \( F \rightarrow \neg \Box K \neg F \) from 7 and 8
10. \( \Box F \rightarrow \Box \neg \Box K \neg F \) 9 Nec and distribution
11. \( \Box F \rightarrow \Diamond \Box K F \) 3 and 10
Theorem 12.
\[ \Box F \rightarrow \Diamond \Box K F \not\models \Diamond \Box K F \lor \Diamond \Box K \neg F \]

Proof.
Consider the model \( M_3 \):

\[
\begin{array}{c}
p \\
\bullet \\
u \\
\rightarrow \\
v \\
\neg p \\
\end{array}
\]

B model \( M_3 \) where \( \Box F \rightarrow \Diamond \Box K F \) holds but \( \Diamond \Box K F \lor \Diamond \Box K \neg F \) fails.

If \( \Box X \) holds at either \( u \) or \( v \) then \( u, v \models X \), hence \( K X, \Box K X, \Diamond \Box K X \) hold in \( M_3 \), hence \( MK \) holds in the model. Since both \( K p \) and \( K \neg p \) fail at all states, neither of \( \Box K p \) and \( \Box K \neg p \) hold at any state. Therefore \( \Diamond \Box K p \) and \( \Diamond \Box K \neg p \) fail at each state, hence \( TMK \) does not hold in \( M_3 \).
B is just one of a family of bi-modal knowability logics.

We can formulate the logics B+SK, B+MK, B+TK, B+TMK.

Each represents a different view of how knowability is supposed to work.

We can also vary the base logic of such systems, e.g. let $K$ be an S5 modality.

All together it appears that the use of a bi-modal logic offers a robust framework for studying knowability.
Thank you!