Introduction to the Bi-modal Analysis of Knowability

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- Is it knowable that 'Syriza formed a coalition government?'
- It depends on how one understands 'knowable'.
- You cannot know false things, but if it were true it would be knowable.
- So we might formalise 'knowable' as:

$$F \rightarrow \Diamond \mathbf{K} F \qquad \text{(Verificationist Knowability – VK)}$$

- For any F, if it is true, then it is knowable.

- Problem: But $F \rightarrow \Diamond \mathbf{K} F$ implies, for any p:

$$p \rightarrow \mathbf{K} p \qquad \text{(Omniscience – OMN)}$$

- Which says that all truths are known.
- This the Church-Fitch "Knowability Paradox"
- It seems we need a better formalization of claims about knowability.

- Analyses of the "knowability paradox" tend to treat **K** as a non-modal operator.
- Almost no approach uses Kripke models to analyse knowability and the paradox.
- We propose a bi-modal approach to knowability statements, and argue that it is sufficiently flexible to give us coherent and useful formalizations of knowability claims.
- Helps uncover what is wrong with the knowability paradox.

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Basic Assumptions

The system B

Knowability Principles
- Stable Knowability
- Monotonic Knowability
- Total Knowability
- Total Monotonic Knowability

Knowability Logics

- Bi-modal logic B is a logic with modalities $\Box/\Diamond$ and $\mathbf{K}$.
- $\mathbf{K}$ is the epistemic modality, informally $\mathbf{K}F$ reads as 'F is known'.
- $\Box/\Diamond$ is an 'investigation' or 'verification' modality, representing the possible courses a process of gathering information might take.
- $\Box F$ reads as 'at all stages of investigation F holds', while $\Diamond F$ means 'at some stage of investigation F holds'
- Hence, $\Box\mathbf{K}F$, says 'at every stage of investigation F is known'.
- $\Diamond\mathbf{K}F$ says that 'at some stage of investigation F is (or becomes) known'

Outline

Axioms:

A1. Axioms of classical propositional logic

A2. $\Box(A \to B) \to (\Box A \to \Box B)$

A3. $\Box A \to A$

A4. $\Box A \to \Box\Box A$

A5. $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$

A6. $\mathbf{K}A \to A$

Inference Rules: *Modus Ponens*, $\Box$ *Necessitation*, $\mathbf{K}$ *Necessitation*

- A model for B is a quadruple $< W, R_\Box, R_\mathbf{K}, V >$.
- W is a set of states.
- $R_\Box$ a transitive and reflexive relation on W.
- $R_\mathbf{K}$ a reflexive relation on W.
- V is a mapping from propositional variables to subsets of W.

We can prove:

Theorem 1.
B is sound and complete with respect to the class of B models.

This is proved via the canonical model construction.

- With the framework we can now say what is wrong with VK.
- If we add $F \to \Diamond \mathbf{K} F$ to B we get the system B+VK.
- B+VK models are B models which have all instances of VK true at each state.
- But now the $R_{\mathbf{K}}$ relation is also *all knowing*, i.e. satisfies:

$$\forall x \forall y (x R_{\mathbf{K}} y \to x = y)$$

- Call a model *omniscient* if all instances of $F \to \mathbf{K} F$ are true at each state.

Theorem 2.
All models of B+VK are omniscient.

Proof.
Assume some state, x, of a B+VK model is such that $x \Vdash F$. Since $R_{\mathbf{K}}$ is all-knowing for any y such that $xR_{\mathbf{K}}y$ we have that $x = y$, and hence every such y is such that $y \Vdash F$, and so $x \Vdash \mathbf{K}F$. Hence $F \rightarrow \mathbf{K}F$ holds at every state of a B+VK model. ∎

Theorem 3.

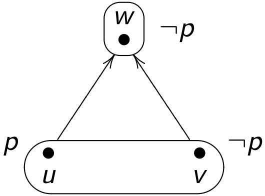
B+VK is complete with respect to the class of B+VK models.

Proof.

As for B except we need to show in addition that $R_{\mathbf{K}}$ is all-knowing. Assume $R_{\mathbf{K}}$ is not all-knowing to derive a contradiction. Let Γ and Δ be maximal consistent sets in the canonical model. Assume $\Gamma R_{\mathbf{K}} \Delta$ but $\Gamma \neq \Delta$; if $\Gamma \neq \Delta$ then there is at least one formula on which they disagree. Assume X is such a formula, and assume $X \in \Gamma$ but $X \notin \Delta$. Since $F \to \Diamond \mathbf{K} F$ is an axiom, $X \to \Diamond \mathbf{K} X \in \Gamma$, and so by the Church-Fitch "knowability paradox" proof $X \to \mathbf{K} X \in \Gamma$. Hence $\mathbf{K} X \in \Gamma$, and so $X \in \Delta$, which is a contradiction.

∎

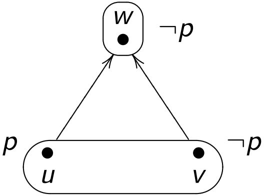
- Consider the following counter-model for $F \to \Diamond \mathbf{K} F$.
- Arrows represent $R_\Box$ and ovals $R_\mathbf{K}$.
- Both are reflexive, and $R_\Box$ is transitive:



B model $\mathcal{M}_1$ in which $F \to \Diamond \mathbf{K} F$ is not valid.

Theorem 4.
Not all instances of $F \to \Diamond \mathbf{K}F$ hold in $\mathcal{M}_1$, in particular $\mathcal{M}_1 \nVdash p \to \Diamond \mathbf{K}p$.

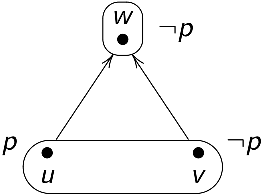
Proof.
At u p is true, but at w and v it is false. Since $R_{\mathbf{K}}$ is reflexive $w \nVdash \mathbf{K}p$, and since $uR_{\mathbf{K}}v$ $u \nVdash \mathbf{K}p$ and $v \nVdash \mathbf{K}p$, hence there is no state $R_{\Box}$-accessible from u where $\mathbf{K}p$ holds, and hence $u \nVdash \Diamond \mathbf{K}p$, so $u \nVdash p \to \Diamond \mathbf{K}p$. Hence not all instances of $F \to \Diamond \mathbf{K}F$ hold in $\mathcal{M}_1$. ∎



B model $\mathcal{M}_1$ in which $F \to \Diamond \mathbf{K}F$ is not valid.

Outline

- Notice that p does not stay true from state u to w.
- Call a proposition *stable*, in a given model, if it satisfies $F \rightarrow \Box F$.
- If p were stable, we would not have a counter-model.
- If p were stable it would be knowable.



B model $\mathcal{M}_1$ in which $F \rightarrow \Diamond \mathbf{K} F$ is not valid.

- To capture this idea we propose:

$$\Box F \to \Diamond \mathbf{K} F. \qquad (\textit{Stable Knowability – SK})$$

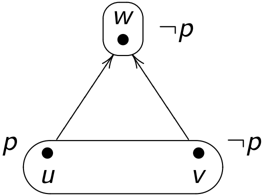
- If F is stably true, then it is knowable, i.e. all stable truths are knowable.
- SK does not entail the omniscience defect

Theorem 5.

$\Box F \rightarrow \Diamond \boldsymbol{K} F \nVdash p \rightarrow \mathbf{K} p$

Proof.

In $\mathcal{M}_1$ if $\Box X$ holds at any state then X holds at w, so $w \Vdash \mathbf{K} X$, hence all states have $\Diamond \mathbf{K} X$, hence all instances of SK hold in $\mathcal{M}_1$, but $v \nVdash p \rightarrow \mathbf{K} p$. ∎



B model $\mathcal{M}_1$ in which $F \rightarrow \Diamond \mathbf{K} F$ is not valid.

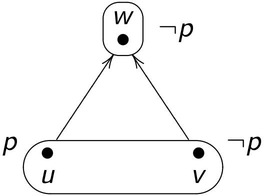
- One might think that knowledge is possible only on the basis of conclusive evidence, i.e. when no possible counter-evidence exists.
- To capture this idea we propose

$$\Box F \to \Diamond \Box \mathbf{K} F. \qquad (\textit{Monotonic Knowability – MK})$$

- If F is stably true then it is knowable indefeasibly.
- MK also does not suffer from the omniscience defect.

Theorem 6.
$\Box F \to \Diamond\Box \mathbf{K} F \not\models p \to \mathbf{K}p$

Proof.
Similar to Theorem 5. If $\Box X$ holds at any state then $\Diamond\Box\mathbf{K}X$ holds at all states, so all instances of MK hold in $\mathcal{M}_1$. ∎



B model $\mathcal{M}_1$ in which $F \to \Diamond\mathbf{K}F$ is not valid.

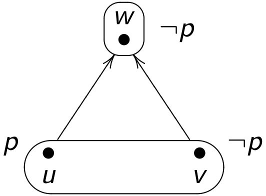
- However, not all propositions are stable.
- Interesting as they are, SK and MK have limited application. (Good for mathematical knowability perhaps?)
- Speaking generally, it seems that saying a proposition is knowable is to say that its truth is decidable (not necessarily formally) by some means or other.
- For non-stable propositions the claims about knowability might be formalised thus:

$$\Diamond \mathbf{K} F \vee \Diamond \mathbf{K} \neg F. \qquad (\textit{Total Knowability} - \textit{TK})$$

- This too does not yield omniscience.

Theorem 7.
$\Diamond \boldsymbol{K}F \vee \Diamond \boldsymbol{K}\neg F \not\Vdash p \to \mathbf{K}p$

Proof.
Again consider $\mathcal{M}_1$. $w \Vdash X \vee \neg X$ so $w \Vdash \mathbf{K}X \vee \mathbf{K}\neg X$, hence $\Diamond\mathbf{K}X \vee \Diamond\mathbf{K}\neg X$ holds at all states, so all instances of TK hold in $\mathcal{M}_1$, but $w \not\Vdash p \to \mathbf{K}p$. ∎



B model $\mathcal{M}_1$ in which $F \to \Diamond\mathbf{K}F$ is not valid.

- Again, one might think that knowledge is indefesible.
- We can formalize this by:

$$\Diamond\Box\mathbf{K}F \vee \Diamond\Box\mathbf{K}\neg F, \quad (\textit{Total Monotonic Knowability} - \textit{TMK})$$

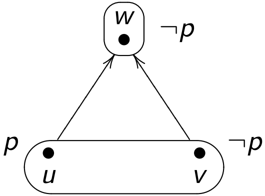
- And again, we can prove TMK does not yield omniscience.

Theorem 8.

$\Diamond\Box\boldsymbol{K}F \vee \Diamond\Box\boldsymbol{K}\neg F \not\Vdash p \to \mathbf{K}p$

Proof.

Similar to Theorem 7. Since $w \Vdash \Box\mathbf{K}X \vee \Box\mathbf{K}\neg X$ and $R_\Box$ is reflexive $w \Vdash \Diamond\Box\mathbf{K}X \vee \Diamond\Box\mathbf{K}\neg X$, hence all instances of $\Diamond\Box\mathbf{K}F \vee \Diamond\Box\mathbf{K}\neg F$ hold at all states. ∎

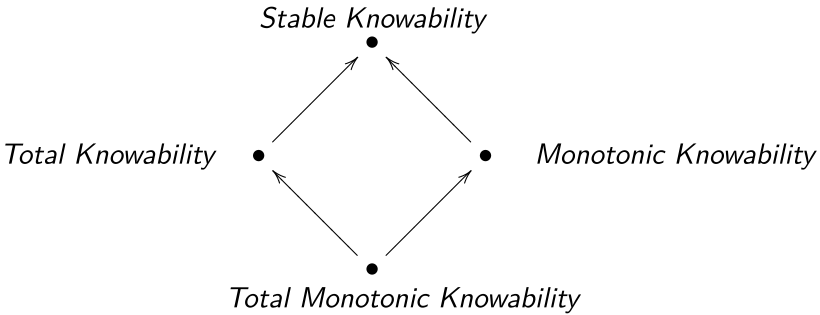


B model $\mathcal{M}_1$ in which $F \to \Diamond\mathbf{K}F$ is not valid.

Outline

- So we see that a bi-modal approach to knowability allows us to formulate different principles.
- It also enables us to compare each of these principles in a rigorous way.
- The following relations hold between the various knowability principles.

"Knowability Diamond"



"Knowability Diamond"

- Arrows represent derivability in B, but not converses.

- Let us prove that TMK is strictly stronger than TK and MK.

Theorem 9.
$\Diamond\Box\mathbf{K}F \vee \Diamond\Box\mathbf{K}\neg F \ \vdash_B \ \Diamond\mathbf{K}F \vee \Diamond\mathbf{K}\neg F$

Proof.
1. $\Diamond\Box\mathbf{K}F \vee \Diamond\Box\mathbf{K}\neg F$
2. $\Box\mathbf{K}F \rightarrow \mathbf{K}F$ Reflexivity
3. $\Box(\Box\mathbf{K}F \rightarrow \mathbf{K}F)$ 2 Nec
4. $\Box\mathbf{K}\neg F \rightarrow \mathbf{K}\neg F$ Reflexivity
5. $\Box(\Box\mathbf{K}\neg F \rightarrow \mathbf{K}\neg F)$ 4 Nec
6. $\Box(\Box\mathbf{K}F \rightarrow \mathbf{K}F) \rightarrow (\Diamond\Box\mathbf{K}F \rightarrow \Diamond\mathbf{K}F)$ from
   $\Box(F \rightarrow G) \rightarrow (\Diamond F \rightarrow \Diamond G)$
7. $\Box(\Box\mathbf{K}\neg F \rightarrow \mathbf{K}\neg F) \rightarrow (\Diamond\Box\mathbf{K}\neg F \rightarrow \Diamond\mathbf{K}\neg F)$ from
   $\Box(F \rightarrow G) \rightarrow (\Diamond F \rightarrow \Diamond G)$
8. $\Diamond\Box\mathbf{K}F \rightarrow \Diamond\mathbf{K}F$ 3, 6 MP
9. $\Diamond\Box\mathbf{K}\neg F \rightarrow \Diamond\mathbf{K}\neg F$ 5, 7 MP
10. $\Diamond\mathbf{K}F \vee \Diamond\mathbf{K}\neg F$ 1, 8, 9 by propositional reasoning.

∎

Theorem 10.

$\Diamond \mathbf{K} F \vee \Diamond \mathbf{K} \neg F \nVdash \Diamond \Box \mathbf{K} F \vee \Diamond \Box \mathbf{K} \neg F$

Proof.

Consider model $\mathcal{M}_2$:

$\mathcal{M}_2$, B model where $\Diamond \mathbf{K} F \vee \Diamond \mathbf{K} \neg F$ holds but $\Diamond \Box \mathbf{K} F \vee \Diamond \Box \mathbf{K} \neg F$ fails

$u \Vdash X \vee \neg X$ so $u \Vdash \mathbf{K} X \vee \mathbf{K} \neg X$ and $u \Vdash \Diamond \mathbf{K} X \vee \Diamond \mathbf{K} \neg X$. Similarly for v, hence at all states TK holds. $u, v \nVdash \Box \mathbf{K} p, \Box \mathbf{K} \neg p$, hence $u \nVdash \Diamond \Box \mathbf{K} p$ and $u \nVdash \Diamond \Box \mathbf{K} \neg p$, hence $\mathcal{M}_2 \nVdash$ TMK.

∎

Theorem 11.

$\Diamond\Box\mathbf{K}F \vee \Diamond\Box\mathbf{K}\neg F \vdash_B \Box F \to \Diamond\Box\mathbf{K}F$

Proof.

1. $\Diamond\Box\mathbf{K}F \vee \Diamond\Box\mathbf{K}\neg F$
2. $\neg\Diamond\Box\mathbf{K}\neg F \to \Diamond\Box\mathbf{K}F$ from 1 by $(F \vee G) \to (\neg F \to G)$
3. $\Box\neg\Box\mathbf{K}\neg F \to \Diamond\Box\mathbf{K}F$ from 2, $\Box\neg$ for $\neg\Diamond$
4. $\mathbf{K}\neg F \to \neg F$ Reflexivity
5. $F \to \neg\mathbf{K}\neg F$ 4 contrapositive
6. $\Diamond F \to \Diamond\neg\mathbf{K}\neg F$ from 5 by Nec and $\Box(F \to G) \to (\Diamond F \to \Diamond G)$
7. $\Diamond F \to \neg\Box\mathbf{K}\neg F$ 6 $\Box\neg$ for $\neg\Diamond$
8. $F \to \Diamond F$ reflexivity
9. $F \to \neg\Box\mathbf{K}\neg F$ from 7 and 8
10. $\Box F \to \Box\neg\Box\mathbf{K}\neg F$ 9 Nec and distribution
11. $\Box F \to \Diamond\Box\mathbf{K}F$ 3 and 10

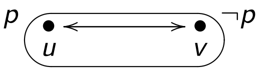
∎

Theorem 12.

$\Box F \to \Diamond\Box \boldsymbol{K} F \nVdash \Diamond\Box \mathbf{K} F \vee \Diamond\Box \mathbf{K}\neg F$

Proof.

Consider the model $\mathcal{M}_3$:



B model $\mathcal{M}_3$ where $\Box F \to \Diamond\Box \mathbf{K} F$ holds but $\Diamond\Box \mathbf{K} F \vee \Diamond\Box \mathbf{K}\neg F$ fails.

If $\Box X$ holds at either u or v then $u, v \Vdash X$, hence $\mathbf{K}X, \Box\mathbf{K}X, \Diamond\Box\mathbf{K}X$ hold in $\mathcal{M}_3$, hence MK holds in the model. Since both $\mathbf{K}p$ and $\mathbf{K}\neg p$ fail at all states, neither of $\Box\mathbf{K}p$ and $\Box\mathbf{K}\neg p$ hold at any state. Therefore $\Diamond\Box\mathbf{K}p$ and $\Diamond\Box\mathbf{K}\neg p$ fail at each state, hence TMK does not hold in $\mathcal{M}_3$.

∎

- B is just one of a family of bi-modal knowability logics.
- We can formulate the logics B+SK, B+MK, B+TK, B+TMK.
- Each represents a different view of how knowability is supposed to work.
- We can also vary the base logic of such systems, e.g. let **K** be an S5 modality.
- All together it appears that the use of a bi-modal logic offers a robust framework for studying knowability.

Thank you!