Semibounded groups

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- Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We call $X \subseteq M^n$ *definable* (in $\mathcal{M}$) if there is a formula $\phi(\bar{x}, \bar{y}) \in \mathcal{L}$ and parameters $\bar{a} \in M^k$, such that
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$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is definable in $\langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$, but not in $\langle \mathbb{R}, <, +, 0 \rangle$.

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- $f : A \subseteq M^m \to M^n$ is *definable* if $\Gamma(f) \subseteq M^m \times M^n$ is definable.

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- Applications: Hrushovski's proof of the function field Modell-Lang conjecture in all characteristics makes use of groups definable in certain structures.

O-minimal structures

Definition (Dries 1982, Pillay-Steinhorn 1986)

A densely linearly ordered structure $\mathcal{M} = \langle M, <, \ldots \rangle$ *is called o-minimal (order-minimal) if every definable subset of* M *is a finite union of open intervals* (a, b), $a, b \in M \cup \{\pm\infty\}$, *and points.*

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- on M we have the $<$-topology,
- on M^n we have the product topology.
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Definition

For every definable $X \subseteq M^n$,

$$\dim(X) = \max\{k : \ X \textit{ contains a } k\textit{-box } I^k \textit{ up to definable bijection, where } I \textit{ is an open interval in } M.\}$$

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Question. (van den Dries, 80's): Is there a structure $\mathcal{N}$ whose class of definable sets lies strictly between that of semilinear and semialgebraic sets?

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If $R = \mathbb{R}$, then $\mathcal{N}$ is the unique structure whose class of definable sets lies strictly between that of $\mathcal{R}_{vect}$ and $\mathcal{R}$.

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Next: semilinear, semibounded, semialgebraic groups.

Groups definable in o-minimal structures

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A definable group G is a topological group.

- G is *(definably) connected* if it contains no proper definable clopen subset.
- G is *(definably) compact* if for every definable $\sigma : (a, b) \to G$, $\lim_{x \to b^-} \sigma(x)$ exists (in G).

Semilinear groups

Let $\mathcal{R}_{vect} = \langle R, <, +, 0, \{x \mapsto rx\}_{r \in R} \rangle$. Let $G_a = \langle [0, a), \oplus, 0 \rangle$ with

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y < a \\ x + y - a & \text{if } x + y \geq a \end{cases}$$

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G_a is 'quotient by a lattice'

$$G_a \cong U_a / \mathbb{Z}a,$$

where $U_a = \bigcup_n [-na, na] \leqslant \langle R, + \rangle$, generated by $[-a, a]$.

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Let G be an abelian, connected, compact group definable in R_{vect} with $\dim(G) = n$. Then

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where

- *U is an open subgroup of $\langle R^n, + \rangle$ generated by a semilinear set*
- *$L = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$ is a subgroup of U, generated by $\mathbb{Z}$-independent $a_1, \ldots, a_n$.*

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- *$L = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$ is a subgroup of U, generated by $\mathbb{Z}$-independent $a_1, \ldots, a_n$.*
- *Moreover, the isomorphism is 'definable'. Namely, there there is a semilinear $S \subseteq U$ (fundamental domain) such that*

$$G \cong_{definably} \langle S, +_L\rangle.$$

Semialgebraic groups - Structure Theorem fails

Let $\langle R, <, +, \cdot, 0, 1\rangle$ be a real closed field. Let $G_b^\times = \langle [1, b), \otimes, 1\rangle$ with:

$$x \otimes y = \begin{cases} xy, & \text{if } xy < b \\ xy/b & \text{if } xy \geq b \end{cases}$$

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But

- $G_b^\times$ is not *definably* isomorphic to G_a, because such an isomorphism would require the existence of an exponential function.
- it is unclear what could be a higher dimensional analogue of (*).

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Example

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- $G_a \times G_b^\times \cong U/L$, where $U = U_a \times G_b^\times$ and $L = \mathbb{Z}(a, 0)$.

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- *$L = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_k$ is a subgroup of U, generated by $\mathbb{Z}$-independent $a_1, \ldots, a_k$.*
- *$k = \operatorname{ldim} G$, the linear dimension of G, is an invariant that counts how semilinear G is.*

General project

Given a structure $\mathcal{N} = \langle \mathcal{M}, P \rangle$,

- define, for every definable set X, an invariant ($\mathcal{M}$-*dimension*) that counts 'how $\mathcal{M}$-definable' X is
- prove structure theorems that 'split' definable groups into their $\mathcal{M}$-definable and P-definable parts.

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For example, $\mathcal{N} = \langle \overline{\mathbb{R}}, \mathbb{Q}^{rc} \rangle$, $\langle \overline{\mathbb{R}}, 2^{\mathbb{Q}} \rangle$, $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.